Non-Perturbative Decay of a Monopole: the Subleading Prefactor

A. K. Monin[†‡]

A. V. Zayakin[†‡]

E-mail: monin@itep.ru, zayakin@itep.ru

[†]Institute for Theoretical and Experimental Physics
117259, Moscow, B. Cheremushkinskaya 25, Russia

[‡] M.V. Lomonosov Moscow State University,
119992, Moscow, Russia

Abstract

The rate of the non-perturbative decay of a 't Hooft–Polyakov monopole in an external electric field into a dyon and a charged fermion is calculated. The subleading semiclassical prefactor is presented for the first time for this process. The leading exponential factor is shown to be in full agreement with the previous results derived in a different technique. Analogous treatment is shown to hold for the two-fermionic decay of the lightest bound state in Thirring model, allowing one to restore the "effective meson–fermion vertex".

1 Introduction

Interest to various aspects of physics of magnetic monopoles has been significant for a long time. Charge quantization [1], baryon decay [2], QCD confinement description [3], duality in gauge theories [4] are just a few examples of important issues associated with monopoles.

The 't Hooft–Polyakov monopole, which is going to be the object of this investigation, is stable; its decay is impossible unless some external field comes into play. On the other hand, there exists a growing interest to spontaneous and induced Schwinger-like decay processes in external fields, as well as to induced vacuum decay processes in various theories, see e.g. [5]. These were the main motivations for us to study the non-perturbatively allowed decay of a monopole into a dyon and a fermion. This paper is organized as follows. In Section 2 some general facts on monopole physics and induced decays are reviewed. We examine the conditions, under which a semiclassical treatment is valid for the considered problem. The corresponding decay rate is calculated in Section 3. The elaborated technique is simplified and applied to bound state decay in Thirring model in Section 4, and the results are summarized in Section 5.

2 Preliminaries

2.1 Monopoles: Non-Perturbative and Non-Local Objects

Since the historic paper by Dirac [1] the question how to incorporate the dynamics of magnetic monopoles into the standard quantum field-theoretical paradigm has been non-trivial. The treatment of monopoles and charges within the same framework is hindered by two obstructions: inapplicability of perturbation theory and non-locality.

Due to Dirac's quantization condition [6], the charge g of a monopole is $g^2 \sim \frac{1}{\alpha} \sim 137$ so that no reasonable perturbation series can be derived according to this parameter, unlike the standard QED perturbation theory in powers of α. Several attempts have been made to elaborate a self-contained QED with monopoles [7], although none of them have been accepted ubiquitously so far.

These two fundamental problems inevitable for the point-like Dirac monopoles arise under a different guise for the 't Hooft–Polyakov monopole. The monopole configuration is *a priori* a solution to classical field equations. It exhibits some properties of a point particle but it cannot be treated as if it were generated by some local field operator[1]. On the other hand, the 't Hooft–Polyakov monopole should be thought of as a kind of semiclassical

[1]In simpler cases, e.g. in sine-Gordon theory, a quantum solitonic object may be written down as an explicitly given non-local field operator (the so-called Mandelstam operator [8]). Monopole creation operator is known in lattice gauge theory [9].

object rather than a quantum particle, as its characteristic size is roughly $1/\alpha$ times greater than its de Broglie wavelength. In a dual theory [4], monopoles correspond to the original gauge bosons which do have local description, however, in the original theory itself, no local description is possible.

The non-perturbative issues of monopole dynamics can be studied via geometric and topological methods, permitting description of dynamics of monopoles and dyons in terms of geodesics of the moduli spaces of solutions to Bogomolny equations [10, 11]. Processes which have an explicit quantum field-theoretical *interpretation* as scattering of monopoles into monopoles or dyons, interaction between them etc. have been shown to take place. However, no quantum field theoretical *model* of these processes exists so far.

String theory suggests describing dyons as (p, q) strings with ends fixed on some D-branes [12]. This description was recently suggested to give rise to the process "gauge boson $\rightarrow$ monopole, dyon" or "monopole $\rightarrow$ dyon, charge" in an external field [5]. The existence of a corresponding vertex in string theory is mentioned to show that there are some attempts to elaborate a local perturbative-like treatment of monopoles. The string vertex is not directly used in the calculations below. However, its existence provides us a heuristic apology to validate introducing an "effective vertex", which, we remind, is absent at perturbation theory level. Besides, a Julia–Zee dyon is heavier than a monopole with the same magnetic charge [13]. There exists a wide class of processes in field theory becoming non-perturbatively allowed should an external field come into play. The obvious example is Schwinger spontaneous e^+e^- pair production in an external electric field, or an analogous process for spontaneous Schwinger-like monopole pair production in a static magnetic field [14].

Another class of non-perturbatively allowed phenomena consists of vacuum decay processes in scalar field theory. The generic case of false vacuum decay in a Higgs-like potential was initially discussed in [15, 16]. There exists a deep similitude between spontaneous Schwinger processes and false vacuum decay. Formally, these two phenomena are identical in $1 + 1$ dimensions [27]. The action S_{cl} of a classical configuration of e^+e^- paths in Euclidean domain contributing to the semiclassical pair creation probability $w \sim e^{-S_{cl}}$ behaves like "$const_1$(volume) $-$ $const_2$(surface)" . The same behaviour is typical for the action of a classical bubble in thin wall approximation, describing, in its turn, vacuum decay probability. This statement is a consequence of fermion–boson duality in $1 + 1$ dimensions, but it can be considered as a hint to a better understanding of more general cases for the both types of processes.

2.2 Induced vs. Spontaneous

The history of false vacuum decay teaches us a lesson that if a process is possible as a spontaneous one, there should exist a related induced one [18, 19]. The same argument works for Schwinger processes. The possibility of an induced Schwinger-type monopole decay was first suggested in [5]. Monopoles were treated as triggers for vacuum decay in a scalar field theory long ago [20]. The present context is different: no underlying scalar field false vacuum is going to be considered.

This interpretation allows one to symbolically introduce an effective vertex "charge–monopole–dyon", although it is non-existent at the level of perturbation theory. Technically, this philosophy means the following. A 't Hooft–Polyakov monopole is considered in what concerns its parameters. As in our previous papers [21, 22] it is treated as a semi-classical object, for which the notion of trajectory is well defined. Only trajectories far larger its size are dealt with, because otherwise the semiclassical approximation is broken down. The trajectory of the monopole is analytically continued into the Euclidean domain, where a correction to its Green function is calculated, yielding the decay rate.

When dealing with Green functions, one is completely aware of the non-existence of an underlying field theory, as mentioned above. So, what do we mean here by Green's function of a monopole or a dyon? These Green's functions stand for an effective one-particle description. True, one is incapable of writing down a quantum field theoretical path integral for the entity being considered, but a 1-particle quantum-mechanical path integral for a particle with given electric charge e and magnetic charge g in an external vector-potential A_μ is meaningful in the semiclassical approximation.

The close relation of the present problem to the issue of false vacuum induced decay has already been pointed out. During calculations, both problems are dealt in a semiclassical technique. Therefore, the structure of the result is similar:

$$\Gamma \sim K e^{-S_{cl}},$$

where the leading exponent behavior is governed by the action on a classical configuration S_{cl}, be it field distribution in field theory or 1-particle trajectory in quantum mechanics; and the subleading pre-exponential factor K generally costs more efforts to be extracted. It contains contributions from Jacobians, arising when integrating out the collective coordinates, and fluctuation determinants.

Basically, two techniques exist for calculating this prefactor: one can either study the fluctuation determinant of the operator describing oscillations around the classical solutions [26] or one can reduce field-theoretical problem to that of 1-particle relativistic quantum mechanics and obtain the prefactor in terms of WKB method [27].

The level of complicatedness of the prefactor calculation depends on the method applied. E.g., the prefactor in Schwinger's derivation of e^+e^- production rate came at the same price with the exponent. On the other hand, when time-dependent field comes into play, it often comes out to be useful to calculate the determinants via Gelfand–Yaglom, Levit–Smilansky [29], or via Riccati equation method [25].

In a paper by one of us (A.K.M.) [21], monopole decay was studied by means of Feynman path integrals in the leading semiclassical approximation. Proof of the existence of a negative mode in the spectrum was also given, but the full fluctuation determinant was not calculated. In our preceding paper this technique was extended to more complicated time-dependent fields [22]. Here a calculation giving the exponential and the pre-exponential factor simultaneously is presented.

3 Monopole in 4D

A monopole with magnetic charge g, mass $M_m \sim M_W/\alpha$ (M_W is the W-boson mass) is considered in a constant external electric field in four-dimensional space-time. We are going to calculate the rate of its decay into a dyon of mass M_d with electric and magnetic charges e, g respectively, and a charged fermion of mass m_e. First the reader is reminded how Green functions can be obtained for an electrically and magnetically charged particle in an external field. Then a "loop correction" is calculated, although this notion has a limited applicability, as commented above.

It has already been commented that the monopole Green's function has got only semiclassical meaning in the proposed approach. This means that one is bound by the requirement for charge-dyon loop to be larger than 't Hooft–Polyakov monopole size. Technically this will imply taking all loop integrals in saddle-point approximation. On the other hand, saddle–point approximation does a good job: it yields the imaginary part of mass correction directly, avoiding the infinite real mass renormalization part[2].

The covariant derivative for a particle with both electric and magnetic charges e and g in external field should look like

$$D_\mu = \partial_\mu + ieA_\mu + ig\tilde{A}_\mu,$$

where vector $\tilde{A}_\mu$ should be chosen so that dual electromagnetic field tensor is equal to $\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu$. This approach was justified by Gibbons and Manton [24].

[2]Monopole mass renormalization due to quantum fluctuations over the classical configuration was discussed in [23]. Mass correction was found out to contain quadratic and logarithmic divergences. After renormalization, finite real non-perturbative mass correction $\delta M_m = -\frac{M_W}{2\pi} \log \frac{M_W}{M_H}$ was found, where M_W, M_H are the W-boson and the Higgs masses respectively. Here mass correction due to a different effect is calculated, namely, induced Schwinger process, not considered in [23]. However, an infinite part of mass correction is implicitly present in our calculation through divergences at $\alpha_i = 0$ in the expression (4) below, avoided by taking the saddle-point approximation

Consider a constant electric field $\mathbf{E} = (0, 0, E)$. Let us choose the vector potential in the form $A_\mu(x) = \frac{E}{2}(-x_3, 0, 0, x_0)$, hence $\tilde{A}_\mu = \frac{E}{2}(0, -x_2, x_1, 0)$. Dirac operator takes the form

$$i\hat{D} - m = i\gamma^\mu D_\mu - m = i\gamma^\mu(\partial_\mu + ieA_\mu + ig\tilde{A}_\mu) - m. \tag{1}$$

A fermionic propagator is given as

$$G_F(y,x) = \left\langle y \left| \frac{1}{m - i\hat{D}} \right| x \right\rangle = -\frac{i}{2}(m + i\hat{D}_y)\int_0^\infty ds e^{i(\frac{m^2}{2}+i\epsilon)s - \frac{1}{2}eEs\gamma^0\gamma^3 - \frac{1}{2}gEs\gamma^1\gamma^2} \langle y|e^{i\frac{s}{2}D^2}|x\rangle, \tag{2}$$

where s is the proper time parameter. Let us define an auxiliary function

$$G^{(0)}(y,x) = \int_0^\infty ds\, e^{i(\frac{m^2}{2}+i\epsilon)s} e^{-\frac{1}{2}eEs\gamma^0\gamma^3} e^{-\frac{1}{2}gEs\gamma^1\gamma^2} \langle y|e^{i\frac{s}{2}D^2}|x\rangle.$$

Terms $i\epsilon$ will be omitted further. One finds

$$G^{(0)}(y,x) = -\frac{i}{32\pi^2}\int_0^\infty \frac{egE^2 ds}{\sinh(\frac{eEs}{2})\sin(\frac{gEs}{2})} e^{i\frac{m^2 s}{2}} e^{-\frac{1}{2}eEs\gamma^0\gamma^3} e^{-\frac{1}{2}gEs\gamma^1\gamma^2} e^{iS},$$

where

$$S = \frac{eE}{4}(y-x)_\parallel^2 \coth\frac{eEs}{2} + \frac{eE}{2}(y_0 x_3 - y_3 x_0) + \frac{gE}{4}(y-x)_\perp^2 \cot\frac{gEs}{2} + \frac{gE}{2}(y_1 x_2 - y_2 x_1),$$

indices $\parallel$ and $\perp$ denote $(0,3)$ and $(1,2)$ components of 4-vector correspondingly. Deforming the s integration contour (roughly speaking, turning it like $s \to is$)[3] and making a transition to Euclidean quantitites like $x_0 \to -ix_0$, one writes down the Euclidean Green function

$$G_E^{(0)}(y,x) = \frac{1}{32\pi^2}\int_0^\infty \frac{egE^2 ds}{\sin(\frac{eEs}{2})\sinh(\frac{gEs}{2})} e^{-\frac{m^2 s}{2}} e^{\frac{1}{2}eEs\gamma^0\gamma^3} e^{\frac{i}{2}gEs\gamma^1\gamma^2} e^{-S_s},$$

where

$$S_s = \frac{eE}{4}(y-x)_\parallel^2 \cot\frac{eEs}{2} - \frac{eE}{2}(y_0 x_3 - y_3 x_0) + \frac{gE}{4}(y-x)_\perp^2 \coth\frac{gEs}{2} - i\frac{gE}{2}(y_1 x_2 - y_2 x_1),$$

all the four-vectors in this expression are supposed to be taken in Euclidean space with the positive overall metric sign; index E will be omitted further. The fermionic propagator thus takes the form

$$G_F(y,x) = (m + \gamma^\mu a_\mu(y,x))\, G^{(0)}(y,x), \tag{3}$$

where

$$\begin{aligned} a_\parallel(y,x) &= \Big(\frac{eE}{2}(y_0 - x_0)\cot\alpha + \frac{eE}{2}(y_3 - x_3),\ \frac{eE}{2}(y_3 - x_3)\cot\alpha - \frac{eE}{2}(y_0 - x_0)\Big), \\ a_\perp(y,x) &= \Big(\frac{gE}{2}(y_1 - x_1)\coth\beta + i\frac{gE}{2}(y_2 - x_2),\ \frac{gE}{2}(y_2 - x_2)\coth\beta - i\frac{gE}{2}(y_1 - x_1)\Big), \end{aligned}$$

[3] As can be seen from (3), the integrand contains term like $\exp(\coth(z))$, posessing essential singularities at $z = \pi i n$. Therefore this transformation is not pure a rotation $s \to is$ but rather a deformation which must avoid traversing the singularity points.

with $\alpha = \frac{eEs}{2}$ and $\beta = \frac{gEs}{2}$.

There are arguments in favour of thinking (0,1)-monopole to be a scalar particle and (1,1)-dyon to be a spin-$\frac{1}{2}$ particle [30]. In brief, they may be recollected as follows: two gauge equivalent descriptions exist for dyons. In one picture, long-range velocity-dependent (non-Coulomb) forces are present between two dyons, but the particles are considered as bosons. In other case, no extraordinary forces are present, but the particles are treated as fermions, if the integer n is odd in Dirac quantization condition $eg = \frac{n}{2}$. Although both descriptions are gauge equivalent, it is more natural to choose the second way of thinking.

The correction to monopole's Green function propagating from $(0,0,0,0)$ to $T = (0,0,0,T)$ in this case may be expressed in terms of Feynman path integrals [21] and reduced to a contraction of Green functions (here we suggest the "vertex" of monopole-dyon-charged fermion interaction to be of the form $\lambda\phi\bar{\psi}\psi$)

$$\delta G_m(T,0) = \lambda^2 \int d^4x d^4y G_m(z,0) \mathrm{tr}[G_e(w,z)G_d(w,z)]G_m(T,w) \tag{4}$$

λ being (an unknown[4]) dimensionless factor, indices m, e, d belonging here and everywhere below to a monopole, a charged fermion and a dyon respectively. The trace above would be proportional to

$$\mathrm{Tr} \equiv \mathrm{tr}(M_d + \hat{a})(\cos\alpha_2 + \gamma^0\gamma^3 \sin\alpha_2)(\cosh\beta_2 + i\gamma^1\gamma^2 \sinh\beta_2)(m_e + \hat{b})(\cos\alpha_1 - \gamma^0\gamma^3 \sin\alpha_1), \tag{5}$$

here a, α_2, $\beta_2 \equiv \frac{g}{e}\alpha_2$ correspond to the dyon propagator and b, α_1 to that of a charged fermion, where one must take into account that the charged fermion has electric charge opposite to that of the dyon and no magnetic charge. Calculating the trace one obtains

$$\begin{aligned} \mathrm{Tr} &= 4\Big(m_e M_d \cosh(\frac{g}{e}\alpha_2)\cos(\alpha_1 - \alpha_2) + \\ &+ \left(\frac{eE}{2}\right)^2 (w-z)^2_{\parallel} \frac{\cosh(\frac{g}{e})\alpha_2}{\sin\alpha_1\, \sin\alpha_2} + \frac{egE^2}{4}(w-z)^2_{\perp} \frac{\cos(\alpha_1-\alpha_2)}{\sinh(\frac{g}{e}\alpha_2)}\Big). \end{aligned}$$

The correction to monopole Green function becomes

$$\begin{aligned} \delta G_m(T,0) &= \frac{1}{2^{18}\pi^4}\lambda^2 e g^3 E^4 \int \frac{d\alpha_1\, d\alpha_2 d\alpha_3\, d\alpha_4\, dz\, dw\, \mathrm{e}^{-(B+S_{\parallel}+S_{\perp})}}{\alpha_1 \sin\alpha_1 \sin\alpha_2 \sinh(\frac{g}{e}\alpha_2)\, \alpha_3\, \alpha_4 \sinh(\frac{g}{e}\alpha_4)\sinh(\frac{g}{e}\alpha_3)} \times \\ &\times \Big(m_e M_d \cosh(\frac{g}{e}\alpha_2)\cos(\alpha_1-\alpha_2) + \left(\frac{eE}{2}\right)^2 (w-z)^2_{\parallel} \frac{\cosh(\frac{g}{e}\alpha_2)}{\sin\alpha_1\, \sin\alpha_2} + \\ &+ \frac{egE^2}{4}(w-z)^2_{\perp} \frac{\cos(\alpha_1-\alpha_2)}{\sinh(\frac{g}{e}\alpha_2)}\Big), \end{aligned} \tag{6}$$

[4]In the next section some arguments will be given for restoring λ form a different calculation in the 2-dimensional case.

where

$$
\begin{aligned}
B &= \tfrac{m_e^2}{eE}\alpha_1 + \tfrac{M_d^2}{eE}\alpha_2 + \tfrac{M_m^2}{eE}(\alpha_3+\alpha_4) \\
S_\parallel &= \tfrac{eE}{4\alpha_4} z_\parallel^2 + \tfrac{eE}{4s}(T-w)_\parallel^2 + \tfrac{eE}{4}(w-z)_\parallel^2(\cot\alpha_1+\cot\alpha_2) \\
S_\perp &= \tfrac{gE}{4} z_\perp^2 \coth(\tfrac{g}{e}\alpha_4) + \tfrac{gE}{4} w_\perp^2 \coth(\tfrac{g}{e}\alpha_3) + \tfrac{eE}{4}\tfrac{(w-z)_\perp^2}{\alpha_1} + \tfrac{eE}{4}(w-z)_\perp^2 \coth(\tfrac{g}{e}\alpha_2) - \\
&\quad -i\tfrac{gE}{2}(w_1 z_2 - w_2 z_1).
\end{aligned}
$$

Integrating out z and w and introducing Feynman variables

$$
\begin{aligned}
\alpha_3 &= Ax, \\
\alpha_4 &= A(1-x),
\end{aligned}
$$

with the Jacobian of the substitution being equal to A, one notes that no dependence on x enters formula (6), thus the x-integration is taken off trivially, after which the correction to Green function becomes

$$
\begin{aligned}
\delta G &= const \int \frac{d\alpha_1\, d\alpha_2\, A\, dA}{\alpha_1 \sin\alpha_1 \sin\alpha_2 \sinh(\frac{g}{e}\alpha_2)} \mathrm{e}^{-\left[\frac{m_e^2}{eE}\alpha_1 + \frac{M_d^2}{eE}\alpha_2 + \frac{M_m^2}{eE}A + \frac{\frac{eE}{4}T^2}{A + \frac{\sin\alpha_1 \sin\alpha_2}{\sin(\alpha_1+\alpha_2)}}\right]} \times \\
&\times \frac{1}{\left[\left(\frac{e}{\alpha_1} + g\cot\frac{g\alpha_2}{e}\right)\sinh\frac{gA}{e} + g\cosh\frac{gA}{e}\right]\left[A(\cot\alpha_1+\cot\alpha_2)+1\right]} \times \\
&\times \Bigg\{ m_e M_d \cosh(\frac{g}{e}\alpha_2)\cos(\alpha_1-\alpha_2) + eE\frac{\cosh(\frac{g}{e}\alpha_2)A}{\sin\alpha_1 \sin\alpha_2[A(\cot\alpha_1+\cot\alpha_2)+1]} + \\
&+ \left(\frac{eET}{2}\right)^2 \frac{\cosh(\frac{g}{e}\alpha_2)}{\sin\alpha_1 \sin\alpha_2[A(\cot\alpha_1+\cot\alpha_2)+1]^2} + \\
&+ egE\frac{\cos(\alpha_1-\alpha_2)\sinh(\frac{g}{e}A)}{\alpha_1 \sinh(\frac{g}{e}\alpha_2)\left[\left(\frac{e}{\alpha_1} + g\cot\frac{g\alpha_2}{e}\right)\sinh\frac{gA}{e} + g\cosh\frac{gA}{e}\right]} \Bigg\}.
\end{aligned} \tag{7}
$$

To integrate over A, the saddle-point approximation is employed. Generally, the saddle-point approximation works for the integrals

$$
\int_0^{+\infty} e^{\nu f(s)} g(s) ds = \sqrt{\frac{2\pi}{-\nu f''(s_0)}} e^{\nu f(s_0)} g(s_0) + \mathrm{O}\left(\frac{1}{\nu}\right) \tag{8}
$$

(s_0 being the minimum point of $f(s)$), when $\nu \to \infty$. In the present case,

$$
\nu f(A) = -\frac{M_m^2}{eE}\left[A + \frac{(eE)^2}{4M_m^2}T^2 \frac{1}{A + const}\right] \tag{9}
$$

satisfies this requirement, as $\nu = \frac{M_m^2}{eE}$ is a large parameter indeed, coefficient $\frac{(eE)^2}{4M_m^2}T^2$ being not infinitesimal, as T may be made large enough for our purposes. In fact, we are going to use the limit $T \to \infty$, so the latter statement is fairly justified.

One may wonder if an analogous saddle-point calculation would be valid for Schwinger pair production. It is easy to see that in case of spontaneous Schwinger process, saddle-point approximation yields the first term in the effective action only up to a multiplicative

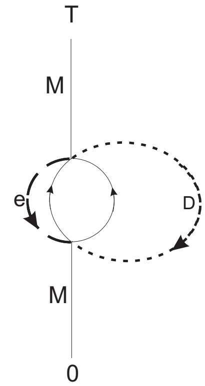

Figure 1: Classical paths in (x_3, x_0) plane with arbitrary winding numbers.

numeric factor, which cannot be neglected. To see why this happens instead of restoring the full answer, let us write down the pair production rate for a uniform external electric field

$$w = \frac{(eE)^2}{4\pi^2} \int_0^{\infty} \frac{ds}{s^2} e^{-\frac{m^2}{eE}s} \cot s \tag{10}$$

and raise the $\sin(s)$ into the exponent. To evaluate $\int_0^{+\infty} e^{\nu f(s)} ds$ via the saddle-point method one should minimize the function $\nu f(s) \equiv -\frac{m^2}{eE}(s - \frac{eE}{m^2} \log \sin s)$, which yields minimum points $s_0^{(n)} = (-1)^{n+1} \sin^{-1} \frac{eE}{m^2} + \pi n \approx \pi n$, $n \in \mathbb{Z}$, $s_0^{(n)} > 0$, if one assumes $\frac{m^2}{eE} \gg 1$. But in fact saddle-point prescription cannot work here because, unlike the previous case, no general large parameter ν can be extracted from $f(s)$, without making the coefficient in front of $\log \sin(s)$ negligible and thus invalidating the method. Indeed one obtains the leading term of the rate, corresponding to the smallest positive $s_0^{(n)}$

$$w_1 = \frac{(eE)^2}{2\sqrt{2}\pi^{7/2}} e^{-\frac{\pi m^2}{eE}}, \tag{11}$$

different from the proper leading term by an overall numerical factor. The common derivation of the sum (11) is through evaluating the integral (10) via the residues in the poles of $\cot s$, yielding the proper $w_1 = \frac{(eE)^2}{4\pi^3} e^{-\frac{\pi m^2}{eE}}$. This is a just a caution against the improper use of the saddle-point approximation. Below we explain why this method nevertheless works for our case.

The saddle point value A_0 in the integral (7) over A is assumed to satisfy $A_0 \gg 1$, so in principle one could consider asymptotics for hyperbolic functions in the form $\sinh \frac{gA}{e} \approx \cosh \frac{gA}{e} \approx \frac{1}{2} \mathrm{e}^{\frac{gA}{e}}$, and raise $\frac{gA}{e}$ to the exponent. However, one should remember that since monopole and dyon are being treated as point-like particles, it is obligatory to consider an external field small enough so that the size of the loop (see Fig. 1) is larger than the size of the monopole.

For such a field it is easy to show that $\frac{M_m^2}{gE} \gg g^2$. So, we must neglect the term gA/e in the exponent compared to $m^2 A/eE$. But we still have gA/e large enough to consider hyperbolic functions $\cosh(\frac{g}{e}A)$ and $\sinh(\frac{g}{e}A)$ approximately equal. Then we obtain the saddle point value for A

$$A_0 = \frac{eET}{2M_m} - \frac{\sin\alpha_1 \sin\alpha_2}{\sin(\alpha_1+\alpha_2)},$$

and the second derivative is

$$\frac{\partial^2 f}{\partial A^2} = \frac{4M_m^3}{(eE)^2 T}.$$

In order to find the monopole mass correction one should know the asymptotic form of the propagator of a scalar particle in an external field. The scalar Euclidean propagator is

$$G_S(T,0) = \frac{1}{16\pi^2} gE \int \frac{ds}{s\,\sinh(\frac{gEs}{2})} \mathrm{e}^{-\frac{M_m^2 s}{2}} \mathrm{e}^{-\frac{T^2}{2s}}.$$

Evaluating it in the saddle-point approximation, taking the limit $T \to \infty$ and using the same suggestion about the small value of the field magnitude, one can find an asymptotic expression for the propagator

$$G_m(T,0) = \frac{1}{16\pi^{3/2}} \frac{gE}{\sqrt{M_m T}} \frac{\mathrm{e}^{-M_m T}}{\sinh\frac{gET}{2M_m}}, \tag{12}$$

and the leading-order (in powers of T) contribution to its variation due to the variation of the monopole mass

$$\delta G_m(T,0) = -\frac{1}{8\sqrt{2}\pi^{3/2}} \delta M_m\, gE \sqrt{\frac{T}{M_m}} \frac{\mathrm{e}^{-M_m T}}{\sinh\frac{gET}{2M_m}}. \tag{13}$$

Comparing this result with the one obtained after integration (7) over A one gets the monopole mass correction

$$\begin{aligned} \mathrm{Im}\,\delta M_m \quad = \quad & -\frac{1}{2^7\sqrt{2}\pi^{3/2}} \frac{\lambda^2 g}{M} \int \frac{d\alpha_1\, d\alpha_2 \mathrm{e}^{-\left(\frac{m_e^2}{eE}\alpha_1 + \frac{M_d^2}{eE}\alpha_2 - \frac{M_m^2}{eE}\frac{\sin\alpha_1\,\sin\alpha_2}{\sin(\alpha_1+\alpha_2)}\right)}}{\alpha_1\,\sinh(\frac{g}{e}\alpha_2)\,\sin(\alpha_1+\alpha_2)\left(\frac{e}{\alpha_1} + g\cot(\frac{g}{e}\alpha_2) + g\right)} \times \\ & \times \left[m_e M_d \cosh\left(\frac{g\alpha_2}{e}\right)\cos(\alpha_1-\alpha_2) + M_m^2 \cosh\left(\frac{g\alpha_2}{e}\right)\frac{\sin\alpha_1 \sin\alpha_2}{\sin^2(\alpha_1+\alpha_2)} \right]. \end{aligned} \tag{14}$$

The terms proportional to E compared to the ones proportional to any bilinear combination of masses have already been neglected here. It was reasonable to leave them out, since such an assumption had already been taken when integrating over A in the saddle-point technique (when not raising to the exponent terms like $\sinh(eA/g)$).

The last step is to integrate over α_1 and α_2 using the saddle point method. Note that the custom integration via methods of the theory of complex variable functions fails, due to an essential non-analyticity in α_1, α_2 present in the expression being studied (roughly speaking, it's like $e^{-1/z}$ in the vicinity of $z = 0$, as can be seen from (14) above). On the contrary, saddle-point approximation remains valid, because all massive parameters are considered to be large compared to $\sqrt{eE}$.

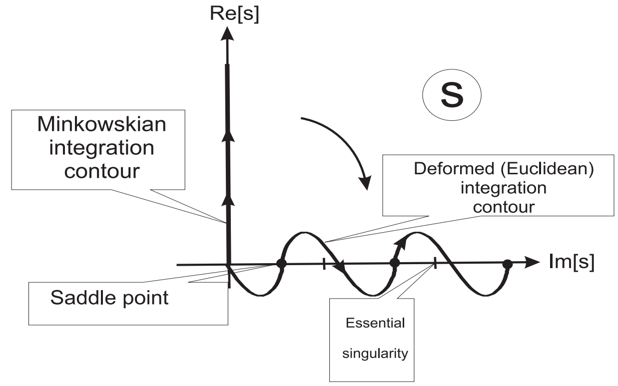

Figure 2: Integration contour for Minkowskian and Euclidean Green Functions.

However, due to the specified essential singularities, complicated deformation of integration contour should be performed. Formula (14) should rather be understood in the following way: one starts in fact, with the Minkowskian Green functions, for which path of integration is directed along the imaginary axis of $z \equiv \alpha_1 + \alpha_2$, being away from essential singularities. In fact, such a contour rotation refers not only to (14), but to (2) and (3) as well. The original Minkowskian Green function was defined with a contour directed along imaginary s axis. When writing down Euclidean Green function (2), one should already have given a prescription for turning the integration contour to the real s axis. How it should have been done, is shown in Fig. 2. Here singularities do not lie on integration path, and saddle-points are passed in the (imaginary) direction prescribed by steepest descent condition. The deformation was performed in the domain of analyticity of the integrand, without traversing the singularities. The integral is dominated by saddle points, and may be evaluated as sum of integrals in the vicinities of each saddle-point. A contour (of real dimension 2) in $\mathbb{C}^2$ for (14) is constructed in a similar way. We do not show it due to high dimensionality.

The function $f(\alpha_1\,,\alpha_2) = \frac{m_e^2}{eE}\alpha_1 + \frac{M_d^2}{eE}\alpha_2 - \frac{M_m^2}{eE}\frac{\sin\alpha_1\,\sin\alpha_2}{\sin(\alpha_1+\alpha_2)}$ is to be minimized. One gets the saddle point values $\theta_i^{(n)}$ for α_i, which come out to be the same as were obtained in [21] by a different method

$$\begin{pmatrix} \theta_1^{\pm(n)} \\ \theta_2^{\pm(m)} \end{pmatrix} = \pm \begin{pmatrix} \cos^{-1}\frac{M_m^2+m_e^2-M_d^2}{2m_e M_m} \\ \cos^{-1}\frac{M_m^2-m_e^2+M_d^2}{2M_d M_m} \end{pmatrix} + \begin{pmatrix} 2\pi n \\ 2\pi m \end{pmatrix} \equiv \begin{pmatrix} \pm\theta_1 + 2\pi n \\ \pm\theta_2 + 2\pi m \end{pmatrix}$$

$$n,\, m \in \mathbb{Z}, \quad \theta_i^{\pm(n)} > 0$$

and the corresponding determinant being

$$\det{}_{ij}\left(\frac{\partial^2 f}{\partial\alpha_i\partial\alpha_j}\right) = -4\frac{\sin^2\theta_1\,\sin^2\theta_2}{\sin^4(\theta_1+\theta_2)}\left(\frac{M_m^2}{eE}\right)^2 = -4\frac{(m_e m_d)^2}{(eE)^2}.$$

One can see that there exists a two-parameter family of local minima of the saddle-point integral. Geometrically, the integer parameters m, n denote multiply-wound classical solutions. The result is a sum over all saddle points. The physical meaning of this sum was discussed in [22]. The semiclassical approximation counts all possible classical sub-barrier trajectories, which are arcs of a circle, θ_i having direct meaning of an angular coordinate on the particle trajectory in the Euclidean plane, taking them with weights $e^{-S^{\pm}_{n,m}}$ given below.

Finally one obtains the mass correction as a sum over winding numbers m, n

$$
\begin{aligned}
\operatorname{Im}\delta M_m &= -\frac{\lambda^2}{8\pi}\frac{eE}{M_m}\times \\
&\times\Bigg\{\sum_{n=0,\,m=0} \mathrm{e}^{-S^+_{n,m}} \frac{\cos^2\left(\frac{\theta_1-\theta_2}{2}\right)}{\sin(\theta_1+\theta_2)\left(\frac{e}{\theta_1+2\pi n}+g\cot\left(\frac{g}{e}(\theta_2+2\pi m)\right)+g\right)}\times \\
&\quad\times\frac{g}{(\theta_1+2\pi n)\tanh\left(\frac{g}{e}(\theta_2+2\pi m)\right)}- \\
&-\sum_{n=1,\,m=1} \mathrm{e}^{-S^-_{n,m}} \frac{\cos^2\left(\frac{\theta_1-\theta_2}{2}\right)}{\sin(\theta_1+\theta_2)\left(\frac{e}{2\pi n-\theta_1}+g\cot\left(\frac{g}{e}(2\pi m-\theta_2)\right)+g\right)}\times \\
&\quad\times\frac{g}{(2\pi n-\theta_1)\tanh\left(\frac{g}{e}(2\pi m-\theta_2)\right)}-\Bigg\},
\end{aligned}
$$

with

$$
\begin{aligned}
S^+_{n,m} &= \tfrac{m_e^2}{eE}(\theta_1+2\pi n)+\tfrac{M_d^2}{eE}(\theta_2+2\pi m)-\tfrac{m_e M_d}{eE}\sin(\theta_1+\theta_2),\\
S^-_{n,m} &= \tfrac{m_e^2}{eE}(2\pi n-\theta_1)+\tfrac{M_d^2}{eE}(2\pi m-\theta_2)+\tfrac{m_e M_d}{eE}\sin(\theta_1+\theta_2).
\end{aligned}
$$

This sum looks rather ugly, however, the contributions of higher winding paths are suppressed by the factor of $\exp[-(\frac{m_e^2}{eE}2\pi n+\frac{M_d^2}{eE}2\pi m)]$. So, for practical calculations only the leading term should be left in the sum. The leading term is the one with "+" and zero winding numbers. It is given as

$$
\operatorname{Im}\delta M_m = -\frac{\lambda^2}{4\sqrt{2}\pi}\frac{eE}{M_m}\,\mathrm{e}^{-S_0}\frac{\cos^2\left(\frac{\theta_1-\theta_2}{2}\right)}{\sin(\theta_1+\theta_2)\left(\frac{e}{\theta_1}+g\cot\left(\frac{g}{e}\theta_2\right)+g\right)}\frac{g}{\theta_1\tanh\left(\frac{g}{e}\theta_2\right)},
$$

with the corresponding value of S_0

$$
S_0=\frac{m_e^2}{eE}\theta_1+\frac{M_d^2}{eE}\theta_2-\frac{m_e M_d}{eE}\sin(\theta_1+\theta_2).
$$

4 Bound state in 2D

If previous considerations are reduced to two dimensions, situation would be technically simpler, because instead of a monopole one would have a free scalar particle, and a fermion–antifermion pair instead of a dyon and a charged fermion. Thus the problem studied above directly reduces to the decay of bound state into a fermion–antifermion pair in Thirring model. For an induced Schwinger process in Thirring model there exists a calculation of

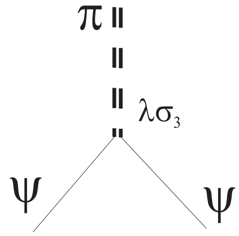

Figure 3: In Thirring model, bound state (π) decays into fermion-antifermion pair, "effective vertex" containing "coupling constant" λ and σ_3 matrix.

the pre-exponential factor in terms of the dual theory by Gorsky and Voloshin [31]. This process is forbidden, as the bound state is lighter than the two fermions, but again, it becomes allowed when an external field is on.

One should note here that the first bound state of massive Thirring model should rather be rendered as a pseudoscalar. Due to duality, bound state in Thirring model corresponds to a special kind of a soliton-antisoliton classical configuration (the so-called "doublet"). Fermionic current j^μ corresponds in the dual picture to the topological current in sine-Gordon

$$\bar{\psi}\gamma^\mu\psi = \epsilon^{\mu\nu}\partial_\nu\phi$$

which can be rewritten as

$$\bar{\psi}\sigma^3\gamma^\mu\psi = \partial^\mu\phi.$$

This suggests that matrix element $\langle 0|\bar{\psi}\sigma^3\psi|\pi\rangle$ is non-zero, σ^3 playing the same role for 2-dimensional case as γ^5 for the 4-dimensional. Thus an "effective vertex" (see Fig. 3) for the considered 2D case should necessarily contain $\sigma^3 = -i\sigma^1\sigma^2$ Pauli matrix. Let us show the final result of the calculation. Here the resummation over winding numbers is done exactly, factors like $\frac{1}{1-e^{-\frac{2\pi\mu^2}{eE}}}$ being a consequence thereof, μ_1 and μ_2 denoting masses of the fermions, which are held arbitrary for the sake of generality:

$$\begin{aligned}\mathrm{Im}\,\delta m \;=\; & \frac{-\lambda^2}{4m\left(1-e^{-\frac{2\pi\mu_1^2}{eE}}\right)\left(1-e^{-\frac{2\pi\mu_2^2}{eE}}\right)\sin(\theta_1+\theta_2)}\left\{\mathrm{e}^{-S_0^+}\left[2\cos^2\left(\frac{\theta_1-\theta_2}{2}\right)-\frac{eE}{\mu_1\mu_2}\frac{1}{\sin(\theta_1+\theta_2)}\right]-\right.\\ & \left.-\,\mathrm{e}^{-S_0^-}\left[2\cos^2\left(\frac{\theta_1-\theta_2}{2}\right)+\frac{eE}{\mu_1\mu_2}\frac{1}{\sin(\theta_1+\theta_2)}\right]\right\},\end{aligned}$$

where

$$\theta_1 = \cos^{-1}\frac{m^2+\mu_1^2-\mu_2^2}{2m\mu_1}$$
$$\theta_2 = \cos^{-1}\frac{m^2-\mu_1^2+\mu_2^2}{2m\mu_2}$$

$$S^+ = \frac{\mu_1^2}{eE}\theta_1 + \frac{\mu_2^2}{eE}\theta_2 - \frac{\mu_1\mu_2}{eE}\sin(\theta_1+\theta_2),$$
$$S^- = \frac{\mu_1^2}{eE}(2\pi-\theta_1) + \frac{\mu_2^2}{eE}(2\pi-\theta_2) + \frac{\mu_1\mu_2}{eE}\sin(\theta_1+\theta_2)$$

Note that λ is an essential parameter here, having dimension of mass. Thirring model calculations for decay of bound state with mass m into two fermions with equal masses μ lead us to

$$\mathrm{Im}\,\delta m = -\frac{\lambda^2}{4m}\frac{\mathrm{e}^{-S_0}}{\sin 2\theta}\left(2 - \frac{eE}{\mu^2}\frac{1}{\sin 2\theta}\right),$$

where

$$\theta = \cos^{-1}\frac{m}{2\mu}$$

(resummation factor $\frac{1}{\left(1-e^{-\frac{2\pi\mu^2}{eE}}\right)^2}$ omitted here).

On the other hand, decay rate in Thirring model in the strong coupling limit (weak coupling limit of Sine-Gordon model) is given [31] as

$$\Gamma = \frac{4g\mu}{\pi^3}e^{-S_0}$$

where g is Thirring coupling constant, $g \gg 1$; μ is the mass of Thirring fermions, S_0 is the classical action. Let us suggest that the external meson is the lightest bound state in the theory, for which in the mentioned limit $m = \frac{\pi^2\mu}{2g}$. It has been obtained by us $\Gamma = 2\mathrm{Im}\delta m = \frac{4\lambda^2 g^2}{\mu\pi^4}e^{-S_0}$ in terms of Thirring model parameters. Comparison of these two formulae yields

$$\lambda = \mu\sqrt{\frac{\pi}{g}}$$

which restores coupling constant λ *in an induced Schwinger process* for the lightest Thirring meson.

5 Conclusion

The pre-exponential subleading asymptotic has been obtained for the non-perturbative monopole decay into a charged fermion and a dyon in 3+1 dimensions, as well as for the decay of a bound state into a fermion-antifermion pair in 1+1 dimensions. These are the main novelties of our work, since these quantities have never been estimated before to this precision. In the two-dimensional case the "effective vertex" $\lambda \sim \frac{\mu}{\sqrt{g}}$ has been restored for the decay of a bound state in Thirring models. Generalization to inhomogeneous fields, thermal field theory as well as to charged fermion decay into monopole-dyon pair are going to be considered as the next problems.

Acknowledgements

Authors are indebted to A. S. Gorsky for suggesting this problem and fruitful discussions, to E. T. Akhmedov, F. V. Gubarev, P. A. Koroteev and A. Yu. Morozov for their useful comments. One of us (A.Z.) would like to thank D. V. Shirkov for his friendly advice and moral support. This work is supported in part by RFBR 04-02-17227 (A.Z.); RFBR 04-01-00646 Grants and Scientific School grant NSh-8065.2006.2 (A.M.).

References

[1] P. A. M. Dirac, Phys. Rev. **74**, 817 (1948).

[2] V. A. Rubakov, IYAI-P-0211

[3] G. 't Hooft, arXiv:hep-th/0010225.

[4] C. Montonen and D. I. Olive, Phys. Lett. B **72** (1977) 117.

[5] A. S. Gorsky, K. A. Saraikin and K. G. Selivanov, Nucl. Phys. B **628** (2002) 270 [arXiv:hep-th/0110178].

[6] P. A. M. Dirac, Proc. Roy. Soc. Lond. A **133** (1931) 60.

[7] L. P. Gamberg and K. A. Milton, Phys. Rev. D **61** (2000) 075013 [arXiv:hep-ph/9910526].

[8] R. Rajaraman, "Solitons And Instantons. An Introduction To Solitons And Instantons In Quantum Field Theory," North Holland Publishing Company, 1982.

[9] J. Frohlich and P. A. Marchetti, Nucl. Phys. B **551** (1999) 770 [arXiv:hep-th/9812004].

[10] M. F. Atiyah and N. J. Hitchin, Phil. Trans. Roy. Soc. Lond. A **315** (1985) 459.

[11] G. W. Gibbons and N. S. Manton, Nucl. Phys. B **274** (1986) 183.

[12] J. Polchinski, "String theory. Vol. 2: Superstring theory and beyond," Cambridge, UK: Univ. Pr. (1998) 531 pp.

[13] B. Julia and A. Zee, Phys. Rev. D **11** (1975) 2227.

[14] I. K. Affleck and N. S. Manton, Nucl. Phys. B **194**, 38 (1982).

[15] I. Y. Kobzarev, L. B. Okun and M. B. Voloshin, Sov. J. Nucl. Phys. **20**, 644 (1975) [Yad. Fiz. **20**, 1229 (1974)].

[16] S. R. Coleman, Phys. Rev. D **15**, 2929 (1977) [Erratum-ibid. D **16**, 1248 (1977)].

[17] C. G. Callan and S. R. Coleman, Phys. Rev. D **16** (1977) 1762.

[18] I. K. Affleck and F. De Luccia, Phys. Rev. D **20** (1979) 3168.

[19] K. B. Selivanov and M. B. Voloshin, JETP Lett. **42** (1985) 422.

[20] P. J. Steinhardt, Nucl. Phys. B **190** (1981) 583.

[21] A. K. Monin, JHEP **0510**, 109 (2005) [arXiv:hep-th/0509047].

[22] A. K. Monin and A. V. Zayakin, JETP Lett. **84** (2006) 5 [arXiv:hep-ph/0605079].

[23] V. G. Kiselev and K. G. Selivanov, Phys. Lett. B **213**, 165 (1988).

[24] G. W. Gibbons and N. S. Manton, Phys. Lett. B **356** (1995) 32 [arXiv:hep-th/9506052].

[25] H. Kleinert, " Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets ," 1450pp. World Scientific, Singapore, 2004.

[26] V. G. Kiselev and K. G. Selivanov, JETP Lett. **39** (1984) 85 [Pisma Zh. Eksp. Teor. Fiz. **39** (1984) 72].

[27] M. B. Voloshin, Yad. Fiz. **42** (1985) 1017 [Sov. J. Nucl. Phys. **42** (1985) 644].

[28] G. V. Dunne and C. Schubert, Phys. Rev. D **72**, 105004 (2005) [arXiv:hep-th/0507174].

[29] S. Levit and U. Smilansky, Annals Phys. **103** (1977) 198.

[30] S. R. Coleman, "The Magnetic Monopole Fifty Years Later," HUTP-82/A032.

[31] A. S. Gorsky and M. B. Voloshin, Phys. Rev. D **73** (2006) 025015 [arXiv:hep-th/0511095].